GEODESICALLY CONVEX AND HYPERCONVEX SUBSETS OF THE PLANE WITH THE MAXIMUM METRIC

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ABSTRACT. A metric space (X, d) is called a geodesic space if for any $p, q \in X$, there exists a path $\alpha : [0, s] \to X$ between p and q, for which $d(p, q) = L(\alpha)$ holds, where $L(\alpha)$ is the length of the path α. $L(\alpha)$ is defined as

$$\sup_{\mathcal{P}} \left\{ \sum_{i=1}^{n} d(\alpha(t_{i-1}), \alpha(t_i)) \right\}$$

over all partitions $P = \{t_0 = 0, t_1, \cdots, t_n = s\}$ of $[0, s]$. Such a path can be reparametrized as $\gamma : [0, d(p, q)] \to X$ satisfying $d(\gamma(t), \gamma(t')) = t' - t$ for $0 \leq t \leq t' \leq d(p, q)$. A path parametrized in this way is called a geodesic between p and q. To give a few examples, consider the unit circle S^1 in the plane with the metric induced from the standard metric of the plane $\mathbb{R}^2$ and choose two antipodal points A and B on it. The distance between the points A and B is 2, but there is no path on S^1 with length realizing this distance. So S^1 with the induced metric is not a geodesic space. If we put however the so-called "shorter arc-length metric" on S^1, then the distance between A and B becomes π and there is a path between A and B with length π (for example, a suitably parametrized half circle). This holds for any two points and S^1 with the shorter arc-length metric becomes a geodesic space. Note that there are two geodesics between the points A and B.

In this talk, geodesics in $\mathbb{R}^2$ with the maximum metric

$$d_\infty((p_1, p_2), (q_1, q_2)) = max\{|p_1 - q_1|, |p_2 - q_2|\}$$

will be determined. A subspace $X \subseteq (\mathbb{R}^2, d_\infty)$ is called geodesically convex if it is a geodesic space with the induced metric i.e. for any two points $p, q \in X$, there exists a geodesic in $(\mathbb{R}^2, d_\infty)$ which is contained in X.

We will also introduce the concept of hyperconvexity and related concepts of injectivity and tight span.

Finally, we will sketch of proof of the following theorem: A nonempty closed and geodesically convex subset of the l_∞ plane $\mathbb{R}^2_\infty$ is hyperconvex and we characterize the tight spans of arbitrary subsets of $\mathbb{R}^2_\infty$ via this property: Given any nonempty $X \subseteq \mathbb{R}^2_\infty$, a closed, geodesically convex and minimal subset $Y \subseteq \mathbb{R}^2_\infty$ containing X is isometric to the tight span $T(X)$ of X.

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