FROM LINEAR SUPERPOSITIONS TO RIDGE FUNCTIONS

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The idea of approximation and representation of complicated multivariate functions by linear superpositions of nice simple functions comes from the need of practical applications. The simplest and at the same time most applicable types of linear superpositions are functions of the form $\sum_{i=1}^{n} g_i(\mathbf{a}^i \cdot \mathbf{x})$. Here g_i are univariate functions, $\mathbf{a}^i$ are nonzero fixed directions in $\mathbb{R}^d$, $d \geq 2$, $\mathbf{x}$ is the variable and the symbol "$\cdot$" stands for the inner product. Summands $g_i(\mathbf{a}^i \cdot \mathbf{x})$ of the above sum are called ridge functions. In other words, a ridge function is a composition of a univariate function with the usual inner product in $\mathbb{R}^d$. These functions arise naturally in various fields. They arise in partial differential equations (where they are called *plane waves*), in computerized tomography, in statistics. Ridge functions are also the underpinnings of many central models in neural network theory, which has become increasing more popular in computer science, statistics, engineering, physics, etc. Finally, these functions are used in modern approximation theory as an effective and convenient tool for approximating complicated multivariate functions. We refer the interested people to the monograph by Allan Pinkus ("Ridge functions", Cambridge University Press 2015) for a detailed study of ridge functions.

Our talk will focus on the following two problems.

1) We are given directions $\mathbf{a}^i$, $i = 1, ..., n$, and a set $X \subset \mathbb{R}^d$. What conditions imposed on a function $f : X \to \mathbb{R}$ are necessary and sufficient for the representation $f(\mathbf{x}) = \sum_{i=1}^{n} g_i(\mathbf{a}^i \cdot \mathbf{x})$?

2) We are given directions $\mathbf{a}^i$, $i = 1, ..., n$. What conditions imposed on a set $X \subset \mathbb{R}^d$ are necessary and sufficient that each function defined on X can be expressed as $\sum_{i=1}^{n} g_i(\mathbf{a}^i \cdot \mathbf{x})$?

First we consider these problems in its most general form, when instead of the above sums of ridge functions, arbitrary linear superpositions are involved. The solutions will lead us to some corollaries, which extend several well known superposition theorems. Then we formulate the obtained general results for sums of ridge functions and analyze in totality some particular cases of special interest.