Behaviour of the Solutions to Ordinary and Delay Differential Equations

I. P. Stavroulakis

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

Email: ipstav@uoi.gr

Visiting Professor at Ankara University

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Abstract

Consider the first order ordinary differential equation of the form

$$x' + px(t) = 0, \quad \text{where } p \text{ is a positive constant.} \tag{A}$$

It is easy to see that all solutions of (A) are of the form

$$x(t) = c\mathrm{e}^{-pt} \quad \text{where } c \text{ is an arbitrary constant.}$$

Next consider the first order delay differential equation of the form

$$x' + px(t-\tau) = 0, \quad \text{where } p \text{ and } \tau \text{ are positive constants.} \tag{B}$$

We show that if

$$p\tau > \frac{1}{\mathrm{e}}, \tag{C}$$

then all solutions of (A) are oscillatory. For example, for the delay differential equation

$$x' + x\big(t - \tfrac{\pi}{2}\big) = 0,$$

we have $p = 1$, $\tau = \frac{\pi}{2}$ and so $p\tau = \frac{\pi}{2} > \frac{1}{\mathrm{e}}$. That is, the condition (C) is satisfied and therefore all solutions oscillate. For example,

$$x(t) = \sin(t) \quad \text{and} \quad x(t) = \cos(t)$$

are oscillatory solutions of (B). Thus, we observe that while all solutions of the equation (without delay)

$$x'(t) + x(t) = 0,$$

are of the form $x(t) = c\mathrm{e}^{-t}$, that is, all are decreasing and tend to zero as $t \to \infty$, however all solutions of the equation

$$x' + x\big(t - \tfrac{\pi}{2}\big) = 0,$$

with delay $\frac{\pi}{2}$ are oscillatory.